Theory of flame-front stability

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A study of the stability of a plane laminar flame front is made. The effects of disturbances on the flame structure are investigated by a small perturbations technique, taking into account the mechanism of diffusion, heat conduction and unsteady combustion. By use of a simplified model of the flame structure, and the assumption that the flame thickness is small compared with the wavelength of disturbances, a formula for the perturbation of the flame propagation velocity is derived. The flame velocity is shown to depend on the curvature of the flame, and on the rates of change of fluid velocities at the flame boundary. From stability analysis it then follows that properties of the mixture, as expressed in terms of the coefficient of heat conductivity and various coefficients of diffusion, play an important role in determining the stability picture. For some estimated values of these parameters the theoretical results are shown to agree with the general trend of the experimentally observed behaviour.

1. Introduction

The problem of stability of a plane flame front in a laminar flow is, from the theoretical point of view, a problem of unsteady dynamics of a fluid in which heat conduction, multicomponent diffusion and complex reactions resulting in heat release take place. A body of experimental evidence on the behaviour of distorted flame fronts is at present available; the theoretical investigations, however, have so far failed to produce a satisfactory explanation of the observed phenomena. The situation is in one respect a very puzzling one: while the experiments show a well-defined behaviour, which seems to be caused by some dominant mechanism, the theory finds it difficult to pin-point such a dominant mechanism. To be more specific let us now briefly outline the problem and summarize the available experimental and theoretical evidence.

We consider a plane flame front which propagates into a combustible mixture with a velocity u_0, the flame propagation velocity. From the point of view of an observer stationed at the moving flame front (figure 1), by approximation three regions of flow can be distinguished: the upstream region, occupied by the combustible mixture, which enters the flame with velocity u_0; the downstream region, occupied by products of combustion, which leave the flame with a different velocity u_f (and at a different temperature); and the flame region itself. In the upstream and downstream regions, heat conductivity, diffusion and reactions can be neglected; while, due to low velocities (u_0 of the order of magnitude of

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30 to 100 cm/sec), compressibility can be neglected as well. Thus, these two regions can be considered to be occupied by ideal fluid. In the flame region, which is usually very thin (of the order of magnitude of a fraction of a millimetre), multicomponent diffusion, heat conductivity and reactions resulting in heat release must be taken into account.

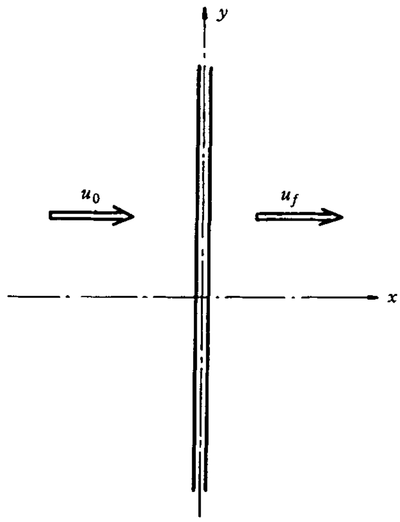

FIGURE 1

Now consider the flame front to be initially slightly distorted. The resulting stability problem was first treated by Landau in 1944 (see Landau & Lifshitz 1953; Emmons 1958). Taking disturbances harmonic in y and linearizing the equations of motion, one can easily find the perturbation fields in the upstream and downstream region. For most observable disturbances the ratio of wavelength to flame thickness will be very large, so that by approximation the flame front may be treated as a discontinuity in the field of fluid. This leads to a simple formulation of the stability problem, provided that a condition is postulated for the propagation velocity of the disturbed flame front. Landau assumed that the disturbances have negligible effect on the flame propagation velocity, hence that, even in disturbed conditions, the normal component of the velocity of fluid relative to the flame front is u_0. He then found the flame to be unstable for all wavelengths.

Let us now turn to experimental evidence. Markstein (1951) studied plane flame fronts for a series of hydrocarbon fuels pre-mixed with air. He found in a great many cases that the flame has a cellular structure: the plane flame front breaks down into many small cells which remain in the plane of the original flame, but which dance around and oscillate. Varying the composition of the mixture, Markstein was able to establish beyond any doubt that almost all fuels produced cellular structure when the mixture was rich, but that no breakdown occurred, and the flame remained plane, when the mixture was lean.†

† The only exception to the rule was produced by methane flames, for which the behaviour was reversed. Of the fuels that were tested, methane happens to be the only one that is considerably lighter than the oxidizer.

If the cellular structure of the flame is interpreted as a result of instability, the conclusion that, for a wide range of fuels, rich mixtures are unstable while lean mixtures are stable must be drawn. The transition point from unstable to stable situations was established by Markstein to be exactly at the stochiometric composition.

Hence it appears that Landau's theory must be modified, so as to take into account the effects of the disturbances on the flame propagation velocity. Markstein proposed that the variation of the flame velocity should be proportional to the ratio of flame thickness to the local radius of curvature of the flame. The proportionality constant remains undetermined in Markstein's analysis and, depending on its value, the effect is stabilizing or destabilizing.

Further theoretical understanding of the flame stability problem can be achieved only if the theory of flame propagation is extended to unsteady and slightly curved flames. But, to date, even the problem of one-dimensional steady-state flame propagation has not been truly solved. The great complexity of the phenomena that take place inside the flame zone, together with partial ignorance about the chemistry of reactions, make the problem very difficult. Nevertheless, based on various assumptions, various theories are available, and, in the words of Spalding (1955), 'no single theory fits all the facts but all fit some of the facts remarkably well'. So there is some hope that, by making reasonable assumptions in the present case, a reasonable theory can be produced. This paper presents an attempt in this direction.

The theory is based on a highly idealized model of the flame structure, which may justifiably be criticized for over-simplification. In many respects the present theory can be described as the simplest possible flame perturbation theory, a counterpart of the stationary Mallard–LeChatelier formula (see Emmons 1958).

As compared with Markstein's hypothesis, the theory shows that the perturbation of the flame velocity is not only proportional to the curvature of the flame, but also to the rate of change of the tangential fluid velocities along the flame, and to the relative acceleration of the flame. The proportionality constants follow from the theory, and are expressed in terms of the physical properties of the burning mixture, such as the coefficients of heat conductivity and various coefficients of diffusion. Stability analysis then shows that rich and lean mixtures indeed behave differently, and that a situation can arise in which the lean mixture is stable for a range of wavelengths, while the rich mixture is unstable.

2. The model of the flame structure

In the present work we adopt the concept of thermal flame propagation. This permits us to subdivide the flame zone by approximation into two regions: a pre-heat region, where the temperature rises mainly because of heat conduction and where the rates of reaction are low; and a burning region, where the rates of reaction are high and where most of the actual combustion takes place (figure 2).

Although little is known about the actual rates of reaction for any complex fuel–oxygen mixture, there are good reasons to assume that the rates of reaction rise very rapidly with temperature. We shall presently assume that the dependence on temperature is so strong that in the pre-heat zone, $\xi < 0$, rates of reaction can be neglected altogether. Moreover, we shall assume that in the burning zone the rates of reactions are very high, so that this zone can be thought to be very thin, and the rate of reaction can be approximated by a constant independent of the temperature. The boundary between the two zones is given by the fictitious ignition temperature T_i which is assumed to be very close to the final

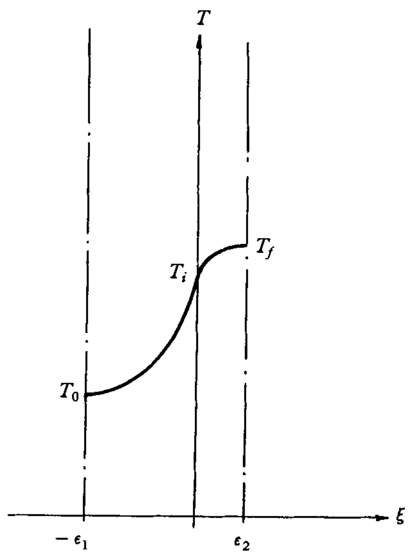

FIGURE 2

flame temperature T_f. Let us now put these assumptions in a more mathematical form. As we shall see very shortly, the parameter that governs the heat conduction in the pre-heat zone is

$$\tau = \frac{5}{2}\frac{knu}{\lambda},$$

where k is the Boltzmann constant, n the number density of the gas, u the velocity, and λ the coefficient of heat conductivity. The inverse of τ measures the width of the pre-heat zone.

Similarly, if we denote the constant rate of reaction in the burning zone by α^*, the inverse of the quantity

$$\frac{\alpha^*}{u_f}$$

measures the width of the burning zone. The fundamental assumption of the present model is

$$\frac{\alpha^*}{u_f\tau} \gg 1.$$

So far we have considered the flame as a temperature wave. This, of course, is only part of the story. The gas that flows through the flame is a mixture of many species. In the pre-heat zone we have predominantly fuel and oxidizer; in the burning zone these species disappear partially or completely and are replaced by products of combustion. There are thus large concentration gradients, and consequently appreciable diffusion. Products of combustion diffuse into the pre-heat zone, while fuel and oxidizer diffuse into the burning zone.

Consider now a slightly curved flame front, which moves with respect to some reference axis (its original plane position) with velocity $\dot{x}_0$, and let the fluid velocities relative to the flame front vary with time. In a co-ordinate system ξ, η attached to the flame front, the flame structure may be assumed to be essentially the one-dimensional stationary flame structure, but with some small perturbations. Let us attempt a qualitative description of these perturbations.

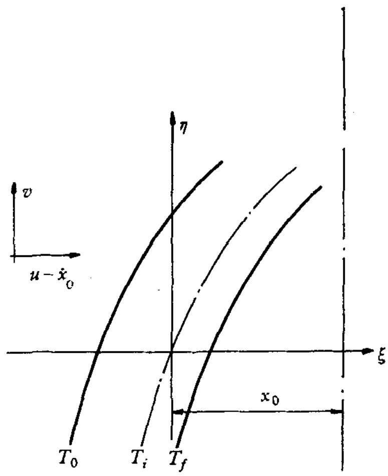

FIGURE 3

The heat conduction is affected by the fact that the flame is no longer plane (and thus one-dimensional) but curved. This is the effect proposed by Markstein; however, there are others too. The heat conduction is also affected by convection due to the perturbation velocities. The result is that locally along the flame the amount of heat needed to raise the temperature in the pre-heat zone to the ignition point is different from that needed in the undisturbed situation.

Next we consider the diffusion processes. In the one-dimensional situation there is a distribution of the relative densities of the various species along the ξ-axis. By the perturbation velocities the species will be convected and redistributed, while there will also be diffusion in the direction tangential to the flame front, due to the flame curvature. The net effect of these processes will be that, as we proceed along the ξ-axis, the ratio of fuel to oxidizer of the mixture which enters the preheat zone, and that of the mixture which enters the burning zone, will no longer be the same. There will be an effective change of composition and thus a perturbation of the amount of heat produced by combustion in the burning zone.

Some of these effects can easily be put in a mathematical form by very simple reasoning (see Eckhaus 1959). However, in order to obtain all the effects correctly, one must proceed from the system of equations that describes the flame structure, develop the perturbation equations and construct their solutions.

3. The system of equations describing the flame structure†

We define a Cartesian co-ordinate system $x-y$, in which the y-axis coincides with the unperturbed position of the flame, and consider the flow of a mixture of species, in which diffusion, heat conduction and reactions take place. Gravitational, viscous and acoustic effects, however, are neglected. The equations that we shall use are adopted from Hirschfelder, Curtiss & Bird (1954) where detailed derivations are given.

To begin with, we have the hydrodynamic equations of continuity and momentum

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v) = 0, \tag{3.1}$$

$$\frac{Du}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \quad \frac{Dv}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y}. \tag{3.2}$$

The velocities u and v are defined as mean mass velocities in the x- and y-directions respectively.

Next we consider separately the species constituting the mixture. The continuity equation for any species $[i]$ is

$$\frac{\partial n_i}{\partial t}+\frac{\partial}{\partial x}[n_i(u+V_{ix})]+\frac{\partial}{\partial y}[n_i(v+V_{iy})] = K_i, \tag{3.3}$$

where n_i is the number-density of species $[i]$, V_{ix} and V_{iy} are diffusion velocities of this species in the x- and y-directions, and K_i represents the number of molecules of species $[i]$ which disappear or are produced per unit of time and volume as a consequence of the combustion process.

The sum of equations (3.3) over all species gives

$$\frac{\partial n}{\partial t}+\frac{\partial}{\partial x}(nS_x)+\frac{\partial}{\partial y}(nS_y) = \Sigma K_i, \tag{3.4}$$

where
$$n = \Sigma n_i, \quad S_x = u+\Sigma\frac{n_i}{n}V_{ix}, \quad S_y = v+\Sigma\frac{n_i}{n}V_{iy}. \tag{3.5}$$

In reality the gas mixture within the flame has a great many components. We idealize this situation by considering a mixture of three species only: fuel, oxidizer, and the products of combustion. Thus, symbolically, we have the reaction $[1]+[2]\rightarrow[3]$. We shall adopt subscript [3] for the products of combustion. With respect to fuel and oxidizer we introduce the following convention: one of the species [1] and [2] disappears completely after combustion. We shall always denote by [1] the species that disappears in the combustion process. Thus, if the mixture is rich [1] will stand for oxygen, if the mixture is lean [1] will denote the fuel.

† We consider here the two-dimensional problem. The final results of the present theory can, however, easily be extended to three-dimensional disturbances, as described by Eckhaus (1959).

The diffusion law for a three component mixture can be written as follows

$$\left(\frac{n_1}{n}\right)\left\{\left(\frac{n_2}{n}\right)\frac{1}{D_{12}}(\mathbf{V}_2-\mathbf{V}_1)+\left(\frac{n_3}{n}\right)\frac{1}{D_{13}}(\mathbf{V}_3-\mathbf{V}_1)\right\} = \operatorname{grad}\left(\frac{n_1}{n}\right), \tag{3.6}$$

$$\left(\frac{n_2}{n}\right)\left\{\left(\frac{n_1}{n}\right)\frac{1}{D_{21}}(\mathbf{V}_1-\mathbf{V}_2)+\left(\frac{n_3}{n}\right)\frac{1}{D_{23}}(\mathbf{V}_3-\mathbf{V}_2)\right\} = \operatorname{grad}\left(\frac{n_2}{n}\right), \tag{3.7}$$

$$n_1 m_1 \mathbf{V}_1 + n_2 m_2 \mathbf{V}_2 + n_3 m_3 \mathbf{V}_3 = 0, \tag{3.8}$$

where D_{ij} is the binary diffusion coefficient between species $[i]$ and $[j]$, and m_i is the molecular weight. By definition $\rho = \Sigma n_i m_i$.

If we now attempt to eliminate the diffusion velocities from the system of equations (3.3), (3.6), (3.7) and (3.8), the resulting differential equations for the number densities become non-linear. It appears thus desirable to introduce some form of approximation. This has been done by Eckhaus (1959). Omitting here the derivation we state the result: in the pre-heat zone, where $K_i = 0$, the approximate laws of diffusion read

$$n_1 \mathbf{V}_1 = -\frac{n^2}{\rho} m_2 D_{12} \operatorname{grad}\left(\frac{n_1}{n}\right) + \frac{n^2}{\rho} m_3 D_3 \left(\frac{n_1}{n}\right)_0 \left[1-\frac{m_2 D_{12}}{m_3 D_{13}}\right] \operatorname{grad}\left(\frac{n_3}{n}\right), \tag{3.9}$$

$$n_3 \mathbf{V}_3 = -n D_3 \operatorname{grad}\left(\frac{n_3}{n}\right), \tag{3.10}$$

$$\frac{1}{D_3} = \left(\frac{n_1}{n}\right)_0 \frac{1}{D_{13}} + \left(\frac{n_2}{n}\right)_0 \frac{1}{D_{23}}, \tag{3.11}$$

where subscript 0 denotes the values at the upstream boundary of the flame front. Using some further approximations concerning the temperature dependence of the diffusion coefficients (see Eckhaus 1959), the following system of equations can be obtained

$$\frac{D}{Dt}\left(\frac{n_3}{n}\right) - D_3 \nabla^2 \left(\frac{n_3}{n}\right) = 0, \tag{3.12}$$

$$\frac{D}{Dt}\left(\frac{n_1}{n}\right) - D_{12} \nabla^2 \left(\frac{n_1}{n}\right) = -D_3 \left(\frac{n_1}{n}\right)_0 \left(1-\frac{D_{12}}{D_{13}}\right) \nabla^2 \left(\frac{n_3}{n}\right). \tag{3.13}$$

The above approximate diffusion equations can be interpreted as follows. Suppose we start by considering species [1] and [2] as one component $[I]$. Then equation (3.12) describes the binary diffusion between $[I]$ and [3]. Having solved this problem we want to find out how the diffusion is distributed between species [1] and [2], and this is described by (3.13).

Proceeding now to the energy equation, in Eckhaus (1959) an approximate form is derived, in which the internal degrees of freedom of the molecules are neglected, the gas velocities are assumed to be low, and the coefficient of heat conductivity is taken to be a constant. The equation then is

$$\lambda \nabla^2 T - \tfrac{5}{2} kn \left[\frac{\partial}{\partial t} + S_x \frac{\partial}{\partial x} + S_y \frac{\partial}{\partial y}\right] T = -Q - \tfrac{5}{2} knT \Sigma K_i, \tag{3.14}$$

where λ is the coefficient of heat conductivity, k is the Boltzmann constant and Q the heat release per unit of volume and time. Also, use has been made of the gas law in the form

$$P = knT. \tag{3.15}$$

We must finally specify the combustion mechanism. Here is where the present model introduces the most crude approximations. Without more justification then, than that the proposed relations are the simplest ones that one can use, we take

in the pre-heat zone, $Q = 0$; $K_i = 0$;

in the burning zone, the reaction is symbolized by the equation

$$a_1[1] + a_2[2] \Rightarrow a_3[3],$$

where a_i denotes the number of molecules of every component that participates in one reaction.

For the heat production we can then write

$$Q = -\frac{q}{a_1} K_1, \tag{3.16}$$

where q is the amount of heat produced in one reaction. We now define α^* to be the fraction of n_1 that participates in the reaction per unit of time. It then follows that

$$K_1 = -\alpha^* n_1, \quad K_2 = -\frac{a_2}{a_1}\alpha^* n_1, \quad K_3 = +\frac{a_3}{a_1}\alpha^* n_1. \tag{3.17}$$

In order to consider perturbations of the flame structure we introduce a new co-ordinate system, attached to the moving flame front

$$\xi = x - x_0(y,t), \quad \eta = y, \quad x_0 = \bar{x}_0(\eta)\, e^{\nu t}. \tag{3.18}$$

Here x_0 measures the distance from the instantaneous position of the ignition point to its position before disturbance was introduced. Thus $\xi = 0$ represents the ignition point in the new co-ordinate system. We now define the perturbation quantities as follows. Let any quantity introduced in this section be $f(x,y,t)$. We write

$$f(x,y,t) = \bar{f}(\xi) + f'(\xi,\eta)\, e^{\nu t}, \tag{3.19}$$

where $\bar{f}$ is the steady-state one-dimensional quantity, and f' is the perturbation; ν can, of course, be complex. We must transform the equations of this section according to (3.18), introduce the various expressions (3.19), and linearize the equations for the perturbation quantities. Instead of summarizing here all the equations obtained in this way, we shall introduce them in the following sections, as the need for them arises.

4. One-dimensional stationary flames

For the theory of perturbations we shall need some results of the stationary one-dimensional flame front. Also, since for stationary one-dimensional flames various theories are available, it is interesting to find out what results the present simple model gives for this case, and compare them with existing flame formulas.

We consider first the pre-heat zone. The energy equation becomes

$$\lambda \frac{d^2\bar{T}}{d\xi^2} - \tfrac{5}{2} k\bar{n}\bar{S}_x \frac{d\bar{T}}{d\xi} = 0. \tag{4.1}$$

The continuity equation (3.4) is

$$\frac{d}{d\xi}(\overline{nS_x}) = 0, \tag{4.2}$$

or

$$\overline{nS_x} = \bar{n}_0\bar{u}_0. \tag{4.3}$$

Thus we introduce the parameter

$$\tau = \tfrac{5}{2}k\frac{\bar{n}_0\bar{u}_0}{\lambda}, \tag{4.4}$$

and obtain the temperature profile in the pre-heat zone

$$\bar{T} = \bar{T}_0 + (\bar{T}_i - \bar{T}_0)\, e^{\tau\xi}. \tag{4.5}$$

Next we consider the burning zone. We introduce one more approximation: we assume that the total number of molecules does not change too much in combustion, $\Sigma K_i = 0$. We then have, for the energy equation,

$$\lambda\frac{d^2\bar{T}}{d\xi^2} - \tfrac{5}{2}k\bar{n}\bar{S}_x\frac{d\bar{T}}{d\xi} + \bar{Q} = 0, \tag{4.6}$$

$$Q = \frac{q}{a_1}\alpha^*\bar{n}_1. \tag{4.7}$$

Result (4.3) still holds in the burning zone. For the continuity equation of species [1] we obtain, from (3.3),

$$\frac{d}{d\xi}[\bar{n}_1(\bar{u} + \bar{V}_1)] = -\alpha^*\bar{n}_1. \tag{4.8}$$

Now, the diffusion process is governed by parameters of the type $\mu = \bar{u}/D_{ij}$, where μ is usually of the same order of magnitude as τ. Our flame model is based on the assumption that $\alpha^*/\bar{u}_f\tau \gg 1$, and thus also $\alpha^*/\bar{u}_f\mu \gg 1$. It follows that the distribution of $\bar{n}_1$ in the burning zone is mainly governed by the rates of reaction, and thus that diffusion can be neglected.

We also make use of the fact that at low velocities the pressure changes across the flame front are negligibly small (see Emmons 1958) so that $\bar{n}\bar{T}$ is constant.

Under these conditions $\bar{n}_1$ can easily be eliminated from (4.6), (4.7) and (4.8), and we obtain

$$\left(\frac{d}{d\xi} + \frac{\alpha^*}{\bar{u}_f}\frac{\bar{T}_f}{\bar{T}}\right)\left[\frac{d\bar{T}}{d\xi} - \tau(\bar{T} - \bar{T}_f)\right] = 0. \tag{4.9}$$

Integrating over the burning zone we now have

$$\left(\frac{d\bar{T}}{d\xi}\right)_{\xi=0} = -\tau(\bar{T}_f - \bar{T}_i) + \frac{\alpha^*}{\bar{u}_f}\bar{T}_f\ln\left(\frac{\bar{T}_f}{\bar{T}_i}\right) + \frac{\alpha^*\tau}{\bar{u}_f}\bar{T}_f\int_0^{\epsilon_2}\frac{\bar{T}_f - \bar{T}}{\bar{T}}\,d\xi. \tag{4.10}$$

Equating the slope of the temperature curve at the ignition point $\xi = 0$ from (4.10) with the solution (4.5) in the pre-heat zone, we obtain

$$\bar{T}_f - \bar{T}_0 = \frac{\alpha^*}{\bar{u}_f\tau}\bar{T}_f\left\{\ln\left(\frac{\bar{T}_f}{\bar{T}_i}\right) + \tau\int_0^{\epsilon_2}\frac{\bar{T}_f - \bar{T}}{\bar{T}}\,d\xi\right\}. \tag{4.11}$$

Since $\overline{T}_f - \overline{T}_i$ is small, the first term between brackets is of the order of magnitude $\overline{T}_f - \overline{T}_i$, while the second one is of the order $\tau\epsilon_2(\overline{T}_f - \overline{T}_i)$, and consequently, much smaller than the first one ($\tau\epsilon_2 \sim \epsilon_2/\epsilon_1$). Thus, retaining only the dominant terms, we have

$$\frac{\alpha^*}{\overline{u}_f\tau}\frac{\overline{T}_f - \overline{T}_i}{\overline{T}_f - \overline{T}_0} = 1, \tag{4.12}$$

which is the Mallard–LeChatelier formula (Emmons 1958).

5. The perturbation of the flame velocity

As discussed in § 2, the perturbations produce two kinds of effects on the flame structure. There is a change of composition of the mixture, resulting from diffusion and convection perturbations, which produces a perturbation of the heat release in the burning zone, and thus a perturbation of the final flame temperature T_f. There are also the convective and curvature effects on the heat conduction process, which result in a perturbation of the amount of heat needed to raise the temperature in the pre-heat zone to the ignition point. By the energy balance, however, this also means a perturbation of the final flame temperature. The net effect of both processes is a perturbation T'_f.

The perturbation of flame propagation velocity is, in the notation of § 3, $(u' - \nu\overline{x}_0)_0$, where subscript zero means that the value at the upstream flame boundary is taken. It seems reasonable to expect that for small perturbations the flame velocity will be proportional to T'_f and therefore governed by a formula of the type

$$\left(\frac{u' - \nu\overline{x}_0}{\overline{u}}\right)_0 = c\frac{T'_f}{\overline{T}_0}, \tag{5.1}$$

where c is some constant which depends on the physical properties of the mixture.

If we consider the Mallard–LeChatelier relation (4.12) and calculate the change of the flame velocity caused by some small change of T_f, we find indeed a formula of the type (5.1), with c given by

$$c = \frac{1}{2}\left\{\left(\frac{\alpha^*}{\overline{u}_f\tau} - 1\right)\frac{\overline{T}_0}{\overline{T}_f - \overline{T}_0} - \frac{\overline{T}_0}{\overline{T}_f}\right\}, \tag{5.2}$$

or, for $\alpha^*/\overline{u}_f\tau \gg 1$,

$$c \simeq \frac{1}{2}\frac{\alpha^*}{\overline{u}_f\tau}\frac{\overline{T}_0}{\overline{T}_f - \overline{T}_0}. \tag{5.3}$$

In Eckhaus (1959) it has been shown that, if the formula for the flame propagation velocity is developed from the complete perturbation equations of § 3, for sufficiently large $\alpha^*/\overline{u}_f\tau$, the result is indeed given by (5.1) and (5.3).

In the present study we shall adopt (5.1) for the flame velocity. Of course, any more sophisticated model of the flame structure would lead to a more complicated flame velocity formula. It is hoped, however, that even though the present model constitutes an over-simplification, the main effects of the perturbations on the flame velocity are reasonably represented by the formula (5.1).

We are now left with the task of evaluating T'_f which in the present theory summarizes all the effects of flame-structure perturbations. This means that we must still consider the complete system of partial differential perturbation equations as defined in § 3, and construct their solutions. However, the task can be considerably simplified by the following consideration. For stability analysis we may assume the ratio of flame thickness to the wavelength of disturbances to be very small. (In a stationary problem of a slightly curved flame front, the ratio of flame thickness to the local radius of curvature will usually be very small.) Thus we may attempt to solve the perturbation equations in terms of power series of the flame thickness parameters. The procedure is particularly simple if only linear terms in flame thickness are retained. The problem can then be solved by an integral method, and the final results are not too complicated.

We consider first the pre-heat zone $\xi < 0$. Linear perturbation of the energy equation (3.14) gives

$$\nabla^2 T' - \tau\left[\frac{\nu}{\bar{S}}T' + \frac{\partial T'}{\partial \xi}\right] = \tau\left[\frac{S_x' - \nu\bar{x}_0}{\bar{S}} + \frac{n'}{\bar{n}}\right]\frac{d\bar{T}}{d\xi} + \frac{d^2\bar{x}_0}{d\eta^2}\frac{d\bar{T}}{d\xi}, \tag{5.4}$$

where use has been made of some results of § 4. Similarly, the continuity equation (3.4) leads to the perturbation equation

$$\frac{\nu}{\bar{S}}\frac{n'}{\bar{n}} + \frac{\partial}{\partial\xi}\left(\frac{n'}{\bar{n}}\right) + \frac{\partial}{\partial\xi}\left(\frac{S_x' - \nu\bar{x}_0}{\bar{S}}\right) + \frac{\partial}{\partial\eta}\left(\frac{S_y'}{\bar{S}}\right) = 0. \tag{5.5}$$

Finally, since the pressure is approximately constant over the flame region, the perturbation of (3.15) leads to

$$\frac{n'}{\bar{n}} = -\frac{T'}{\bar{T}}. \tag{5.6}$$

Combining the above equations we have

$$\frac{\partial^2 T'}{\partial\xi^2} = \tau\frac{\partial}{\partial\xi}\left[\bar{T}\frac{S_x' - \nu\bar{x}_0}{\bar{S}}\right] + \tau\bar{T}\frac{\partial}{\partial\eta}\left(\frac{S_y'}{\bar{S}}\right) + \frac{d^2\bar{x}_0}{d\eta^2}\frac{d\bar{T}}{d\xi} - \frac{\partial^2 T'}{\partial\eta^2}. \tag{5.7}$$

We now integrate (5.7) over the pre-heat zone, and, making use of the condition that temperature perturbations must disappear at $\xi = -\epsilon_1$, we obtain

$$\frac{1}{\tau}\left(\frac{\partial T'}{\partial\xi}\right)_i = \bar{T}_i\left[\frac{S_x' - \nu\bar{x}_0}{\bar{S}}\right]_i - \bar{T}_0\left[\frac{S_x' - \nu\bar{x}_0}{\bar{S}}\right]_0 + \int_{-\epsilon_1}^{0}\bar{T}\frac{\partial}{\partial\eta}\left(\frac{S_y'}{\bar{S}}\right)d\xi$$
$$+\frac{1}{\tau}(\bar{T}_i - \bar{T}_0)\frac{d^2\bar{x}_0}{d\eta^2} - \frac{1}{\tau}\int_{-\epsilon_1}^{0}\frac{\partial^2 T'}{\partial\eta^2}d\xi. \tag{5.8}$$

Note that the last term on the right-hand side is of second order in flame thickness, and can thus be neglected.

Consider now the burning zone. Identical derivation leads to

$$\frac{1}{\tau}\left(\frac{\partial T'}{\partial\xi}\right)_i = \frac{1}{\tau}\left(\frac{\partial T'}{\partial\xi}\right)_f - \bar{T}_f\left[\frac{S_x' - \nu\bar{x}_0}{\bar{S}}\right]_f + \bar{T}_i\left[\frac{S_x' - \nu\bar{x}_0}{\bar{S}}\right]_i$$
$$+\frac{1}{\tau\lambda}\int_0^{\epsilon_2}Q'd\xi - \int_0^{\epsilon_2}\bar{T}\frac{\partial}{\partial\eta}\left(\frac{S_y'}{\bar{S}}\right)d\xi - \frac{1}{\tau}(\bar{T}_f - \bar{T}_i)\frac{d^2\bar{x}_0}{d\eta^2} + O(\epsilon^2). \tag{5.9}$$

Since outside the flame region the temperature fluctuations will only be convected downstream, we have

$$\left(\frac{\partial T'}{\partial\xi}\right)_f = -\frac{\nu}{\bar{u}_f}T_f'. \tag{5.10}$$

Also, we make use of the fact that diffusion velocities must vanish at the flame boundaries. Combining (5.8) and (5.9) we then have

$$\frac{1}{\tau\lambda}\int_0^{\epsilon_2} Q' d\xi = \overline{T}_f\left(\frac{u'-\nu\overline{x}_0}{\overline{u}}\right)_f - \overline{T}_0\left(\frac{u'-\nu\overline{x}_0}{\overline{u}}\right)_0 + \frac{\nu}{\overline{u}_f\tau}T'_f + \int_{-\epsilon_1}^{\epsilon_2}\overline{T}\frac{\partial}{\partial\eta}\left(\frac{S'_y}{\overline{S}}\right)d\xi + \frac{1}{\tau}(\overline{T}_f - \overline{T}_0)\frac{d^2\overline{x}_0}{d\eta^2} + O(\epsilon^2). \quad (5.11)$$

We next compute the perturbation of the heat-release Q'. From (3.3), (3.16) and (3.17) we have the linear perturbation equation

$$Q' = -\frac{q}{a_1}\left\{\nu(n_1)' + \frac{\partial}{\partial\xi}[n_1(u-\nu\overline{x}_0+V_{1x})]' + \frac{\partial}{\partial\eta}[(\overline{n}_1)v' + (n_1V_{1x})']\right\}, \quad (5.12)$$

where q can easily be found to be

$$q = \tau\lambda a_1\frac{\overline{T}_f - \overline{T}_0}{(\overline{n_1 u})_0}. \quad (5.13)$$

The total heat-release perturbation is thus

$$\frac{1}{\tau\lambda}\int_0^{\epsilon_1} Q' d\xi = -\frac{\overline{T}_f - \overline{T}_0}{(\overline{n_1 u})_0}\left\{\nu\int_0^{\epsilon_2} n'_1 d\xi - [n_1(u-\nu x_0+V_{1x})]'_i + \int_0^{\epsilon_2}\frac{\partial}{\partial\eta}[\overline{n}_1 v' + (n_1V_{1y})']\,d\xi\right\}. \quad (5.14)$$

We can also integrate the continuity equation for species [1] over the pre-heat zone to obtain

$$[n_1(u-\nu x_0+V_{1x})]'_i = (\overline{n}_1)_0(u'-\nu x_0)_0 - \nu\int_{-\epsilon_1}^0 n'_1 d\xi - \int_{-\epsilon_1}^0 \frac{\partial}{\partial\eta}[\overline{n}_1 v' + (n_1 V_{1y})']\,d\xi. \quad (5.15)$$

Combining (5.11), (5.14) and (5.15), and using the integrated form of (5.5) we find

$$\left(1+\frac{\nu}{\overline{u}_f\tau}\right)T'_f = -\nu\left\{\frac{\overline{T}_f - \overline{T}_0}{(\overline{n_1 u})_0}\int_{-\epsilon_1}^{\epsilon_2} n'_1 d\xi + \overline{T}_f\int_{-\epsilon_1}^{\epsilon_2}\frac{T' d\xi}{\overline{S}\overline{T}}\right\} - \frac{1}{\tau}(\overline{T}_f - \overline{T}_0)\frac{d^2\overline{x}_0}{d\eta^2}$$
$$- (\overline{T}_f - \overline{T}_0)\overline{\left(\frac{n}{n_1}\right)}_0\int_{\epsilon_1}^{\epsilon_2}\left[\overline{\left(\frac{n_1}{n}\right)} - \overline{\left(\frac{n_1}{n}\right)}_0\right]\frac{\partial v'}{\partial\eta}\frac{d\xi}{\overline{S}} + \int_{-\epsilon_1}^{\epsilon_2}(\overline{T}_0 - \overline{T})\frac{\partial v'}{\partial\eta}\frac{d\xi}{\overline{S}}$$
$$- (\overline{T}_f - \overline{T}_0)\frac{1}{(\overline{n_1 u})_0}\int_{-\epsilon_1}^{\epsilon_2}\frac{\partial}{\partial\eta}(n_1V_{1y})'\,d\xi + \frac{1}{(\overline{n_1 u})_0}\int_{-\epsilon_1}^{\epsilon_2}(\overline{T}_f - \overline{T}_0)\frac{\partial}{\partial\eta}\Sigma\,(n_iV_{iy})'\,d\xi$$
$$+ O(\epsilon^2). \quad (5.16)$$

T'_f is expressed by equation (5.16) in terms of integrals over the flame region. Since we wish to retain in the expression for T'_f only linear terms in flame thickness, it is sufficient to retain in the integrands only terms independent of the flame thickness. Moreover, since $\epsilon_2 \ll \epsilon$, we may neglect the contribution of the burning zone to the integrals.

Consider the tangential momentum equation (3.2). The perturbation equation, when integrated from $\xi = -\epsilon_1$ to an arbitrary point inside the flame region, gives

$$v' + \overline{u}\frac{d\overline{x}_0}{d\eta} = \left[v' + \overline{u}\frac{dx_0}{d\eta}\right]_0 + O(\epsilon). \quad (5.17)$$

Similarly, for the tangential diffusion velocities, we find from (3.9) and (3.10) that

$$(n_i V_{iy})' = -\overline{n_i V_i}\frac{d\bar{x}_0}{d\eta} + O(\epsilon). \tag{5.18}$$

The magnitude $\overline{n_i V_i}$ follows from the stationary form of (3.3):

$$\overline{n_i V_i} = (\overline{n_i u})_0 - \overline{n_i u}. \tag{5.19}$$

Introducing these results into (5.16) and using (5.1), we finally obtain

$$\left(\frac{u'-\nu\bar{x}_0}{\bar{u}}\right)_0 = -\nu c\left\{\frac{\overline{T}_f-\overline{T}_0}{(\overline{n_1 u})_0}\int_{-\epsilon_1}^{0} n_1' d\xi + \overline{T}_f\int_{-\epsilon_1}^{0} T'\frac{d\xi}{\overline{S}\overline{T}}\right\}$$
$$-c\frac{\partial}{\partial\eta}\left[v'+\bar{u}\frac{d\bar{x}_0}{d\eta}\right]_0\left\{\int_{-\epsilon_1}^{0}\frac{\overline{T}-\overline{T}_0}{\overline{T}}d\xi - (\overline{T}_f-\overline{T}_0)\overline{\left(\frac{n}{n_1}\right)_0}\int_{-\epsilon_1}^{0}\left[\overline{\left(\frac{n_1}{n}\right)}-\overline{\left(\frac{n_1}{n}\right)_0}\right]\frac{d\xi}{\overline{S}}\right\}$$
$$+O(\epsilon^2). \tag{5.20}$$

Thus for the evaluation of the flame-velocity perturbation we need for the first group of terms the solutions of the perturbation diffusion and heat-conduction problems. However, we can neglect in these solutions terms of the order of magnitude of the flame thickness. For the evaluation of the second group of terms we need only the solutions of the stationary one-dimensional flame structure.

6. The diffusion problem

Solution of the diffusion equations (3.12) and (3.13) is a straightforward procedure which has been performed in some detail in Eckhaus (1959). Omitting here the calculations, we shall state the results.

Introduce the following diffusion parameters

$$\mu_1 = \frac{\bar{u}_0}{D_{12}}, \quad \mu_3 = \frac{\bar{u}_0}{D_3}, \tag{6.1}$$

$$\Omega = \frac{D_3}{D_{13}}\frac{D_{13}-D_{12}}{D_3-D_{12}}. \tag{6.2}$$

The solution of the one-dimensional steady-state problem is then

$$\overline{\left(\frac{n_3}{n}\right)} = \overline{\left(\frac{n_3}{n}\right)_f} e^{\mu_3\xi}, \tag{6.3}$$

$$\overline{\left(\frac{n_1}{n}\right)} = \overline{\left(\frac{n_1}{n}\right)_0}\left\{1-e^{\mu_1\xi}-\Omega\overline{\left(\frac{n_3}{n}\right)_f}(e^{\mu_3\xi}-e^{\mu_1\xi})\right\}, \tag{6.4}$$

where $\overline{(n_3/n)_f}$ is the relative concentration of the products of combustion in the burned gas.

For the perturbation problem we find that

$$\left(\frac{n_3}{n}\right)' = \left(\frac{u'-\nu\bar{x}_0}{\bar{u}}\right)_0\overline{\left(\frac{n_3}{n}\right)_f}\mu_3\xi e^{\mu_3\xi}+O(\epsilon), \tag{6.5}$$

$$\left(\frac{n_1}{n}\right)' = -\left(\frac{u'-\nu\bar{x}_0}{\bar{u}}\right)_0\overline{\left(\frac{n_1}{n}\right)_0}\left\{e^{\mu_1\xi}+\Omega\overline{\left(\frac{n_3}{n}\right)_f}\left(\frac{\mu_3}{\mu_1}e^{\mu_3\xi}-e^{\mu_1\xi}\right)\right\}\mu_1\xi+O(\epsilon). \tag{6.6}$$

7. The heat-conduction problem

The stationary one-dimensional solution (4.5) we have found already. The perturbation problem is somewhat more complicated. We have a coupled system of equations, (5.4) and (5.5). Eliminating terms involving S'_x we can write the governing equation in the form

$$\frac{\partial}{\partial\xi}\left[\frac{1}{\overline{T}}\left(\frac{\partial T'}{\partial\xi}-\tau T'\right)\right]-\frac{\nu\tau}{\overline{u}_0}\frac{\overline{T}_0}{\overline{T}^2}T'+\frac{\partial}{\partial\xi}\left(\frac{1}{\overline{T}}\int_{-\epsilon_1}^{\xi}\frac{\partial^2 T'}{\partial\eta^2}d\xi'\right)=\tau\frac{\partial}{\partial\xi}\left[\frac{1}{\overline{T}}R(\xi)\right], \tag{7.1}$$

$$R(\xi)=(\overline{T}-\overline{T}_0)\left(\frac{u'-\nu\overline{x}_0}{\overline{u}}\right)_0+\frac{\overline{T}}{\overline{S}}\int_{-\epsilon_1}^{\xi}\frac{\partial S'_y}{\partial\eta}d\xi'-\overline{T}\int_{-\epsilon_1}^{\xi}\frac{1}{\overline{S}}\frac{\partial S'_y}{\partial\eta}d\xi'+\frac{1}{\tau}(\overline{T}-\overline{T}_0)\frac{d^2\overline{x}_0}{\partial\eta^2}. \tag{7.2}$$

The relative order of magnitude of the various terms can be analysed by the transformation $\xi^*=\tau\xi$. We then have

$$\frac{\partial}{\partial\xi^*}\left[\frac{1}{\overline{T}}\left(\frac{\partial T'}{\partial\xi^*}-T'\right)\right]-\frac{\nu}{\overline{u}_0\tau}\frac{\overline{T}_0}{\overline{T}^2}T'+\frac{1}{\tau^2}\frac{\partial}{\partial\xi^*}\left(\frac{1}{\overline{T}}\int_{-\epsilon_1}^{\xi^*}\frac{\partial^2 T'}{\partial\eta^2}d\xi'\right)=\frac{\partial}{\partial\xi^*}\left[\frac{1}{\overline{T}}R(\xi^*)\right], \tag{7.3}$$

$$R(\xi^*)=(\overline{T}-\overline{T}_0)\left[\frac{u'-\nu\overline{x}_0}{\overline{u}}\right]_0+\frac{1}{\tau}\left\{\frac{\overline{T}}{\overline{S}}\int_{-\epsilon_1}^{\xi^*}\frac{\partial S'_y}{\partial\eta}d\xi'-\overline{T}\int_{-\epsilon_1}^{\xi^*}\frac{1}{\overline{S}}\frac{\partial S'_y}{\partial\eta}d\xi'+(\overline{T}-\overline{T}_0)\frac{d^2\overline{x}_0}{\partial\eta^2}\right\}. \tag{7.4}$$

We are looking for some form of expansion of T' in terms of the inverse powers of τ (the flame-thickness parameter), and, as discussed in §5, it is sufficient for the present purpose to evaluate only the term of T' which is independent of τ. This suggests that we should neglect directly in (7.3) and (7.4) all terms of order τ^{-1} and τ^{-2}. The resulting equation is a very simple one, but unfortunately it does not lead to correct results. If we solve it, and consider the solution as a first step in an iteration process, we find that the next approximation to T', which is of the order τ^{-1}, does not satisfy the boundary condition that T' should vanish in the vicinity of the outer flame boundary. A more careful approach is thus needed. We must consider a more complete form of (7.3) and (7.4), construct a solution with a proper behaviour, and evaluate the term of T' that is of interest as a limit of the solution for $\tau\to\infty$.

Consider now the equation

$$\frac{\partial}{\partial\xi^*}\left[\frac{1}{\overline{T}}\left(\frac{\partial T'}{\partial\xi^*}-T'\right)\right]-\frac{\nu}{\overline{u}_0\tau}\frac{\overline{T}_0}{\overline{T}^2}T'=\left(\frac{u'-\nu\overline{x}_0}{\overline{u}}\right)_0\frac{d}{d\xi^*}\left(\frac{\overline{T}-\overline{T}_0}{\overline{T}}\right). \tag{7.5}$$

This may seem a somewhat inconsistent approximation, since in (7.3) we have retained terms of the order τ^{-1}, while in the definition (7.4) of $R(\xi^*)$ we have neglected them. However, $R(\xi^*)$, which is the forcing function of the problem, is not responsible for the behaviour of the solution. If more terms of $R(\xi^*)$ are retained, they prove to contribute only to the labour; in the final results their contribution can be neglected. In the present presentation they have therefore already been neglected at the outset.

Introduce now a new unknown function

$$T'=\frac{\overline{u}_0\tau}{\nu}\left[\frac{u'-\nu\overline{x}_0}{\overline{u}}\right]_0(\overline{T}-\overline{T}_0)(T^*-1). \tag{7.6}$$

Equation (7.5) then becomes

$$\frac{\partial}{\partial \xi^*}\left[\frac{\bar{T}-\bar{T}_0}{\bar{T}}\frac{\partial T^*}{\partial \xi^*}\right]-\frac{\nu}{\bar{u}_0 \tau}\frac{\bar{T}_0}{\bar{T}}\frac{\bar{T}-\bar{T}_0}{\bar{T}}T^* = 0. \tag{7.7}$$

The most convenient form can be obtained by putting

$$T^* = e^{\sigma\xi^*}G(\xi^*), \quad \sigma = -\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\left(1+4\frac{\nu}{\bar{u}_0\tau}\right)}, \tag{7.8}$$

and transforming the co-ordinates by

$$z = \frac{\bar{T}}{\bar{T}_0}. \tag{7.9}$$

We then obtain

$$z(1-z)\frac{d^2G}{dz^2}-[1+(z+\sigma)z]\frac{dG}{dz}-\sigma^2 G = 0, \tag{7.10}$$

which is the standard form of the hypergeometric equation. From Erdelyi (1953) we find the two independent solutions

$$G(z) = \mathscr{C}_1 F[\sigma,\sigma;2\sigma+2;1-z]+\mathscr{C}_2 z^2 F[\sigma+2,\sigma+2;3;z], \tag{7.11}$$

where F is the hypergeometric function. We apply now the boundary condition that the temperature perturbation must vanish near the outer flame boundary; thus $T' \to 0$ when $z \to 1$.

Investigation of the behaviour of the two hypergeometric functions shows that the condition can only be satisfied if $\mathscr{C}_2 = 0$. To determine $\mathscr{C}_1$ we need a boundary condition at the ignition point, where we can have a perturbation of the ignition temperature T_i'. But since the ignition temperature is a fictitious magnitude, it is difficult to postulate a condition for its perturbation. In fact, this is one of the points in which the over-simplification of the flame model brings the theory into trouble. We may, however, reason as follows. Physically, the ignition temperature represents an approximation of the fact that rates of reaction are very low at low temperature and increase rapidly at a sufficiently high temperature. Therefore the question is really whether the perturbations of the flame structure influence rates of reaction so that, in comparison with unperturbed conditions, this rapid rise takes place at a somewhat different temperature. For a thermally propagating flame there seems to be not much reason for effects of this kind. We shall therefore assume that $T_i' = 0$. With this, somewhat speculative, boundary condition, we find that

$$\frac{1}{\mathscr{C}_1} = F[\sigma,\sigma;2\sigma+2;1-z_i], \quad z_i = \frac{\bar{T}_i}{\bar{T}_0}. \tag{7.12}$$

The temperature distribution in the pre-heat zone is now completely specified.

8. Final results of the flame propagation velocity

The results of §§ 6 and 7 must now be substituted into (5.20) and the integrals must be evaluated. Also, the integrals involving T' must be expanded in terms of inverse powers of τ. The necessary calculations have been performed in detail by Eckhaus (1959). We shall here summarize the results.

The perturbation formula for the flame propagation velocity is

$$\left(\frac{u'-v\bar{x}_0}{\bar{u}}\right)_0 = -c\left\{\frac{\nu}{\bar{u}_0\tau}\left(\frac{u'-v\bar{x}_0}{\bar{u}}\right)_0 A_1+\frac{1}{\tau}\frac{\partial}{\partial\eta}\left(\frac{v'}{\bar{u}}+\frac{d\bar{x}_0}{\partial\eta}\right)_0 A_2\right\}+O(\epsilon^2). \tag{8.1}$$

The coefficients A_1 and A_2 can be written as follows

$$A_1 = -\frac{r-1}{r}(K_1+K_2)+K_3, \quad A_2 = \frac{r-1}{r}(K_1+K_2). \tag{8.2}$$

Here, r denotes the ratio of the final flame temperature to the temperature of the unburned mixture; thus

$$r = \bar{T}_f/\bar{T}_0.$$

Furthermore,

$$K_1 = -\tau\bar{T}_f\int_{-\infty}^{0}(e^{\mu_1\xi}-e^{\tau\xi})\frac{d\xi}{\bar{T}}, \tag{8.3}$$

$$K_2 = -\tau\bar{T}_f\overline{\left(\frac{n_3}{n}\right)_f}\,\Omega\int_{-\infty}^{0}(e^{\mu_3\xi}-e^{\mu_1\xi})\frac{d\xi}{\bar{T}}, \tag{8.4}$$

where

$$\bar{T} = \bar{T}_0+(\bar{T}_f-\bar{T}_0)\,e^{\tau\xi}.$$

Finally,

$$K_3 = \frac{3}{4}\frac{r-1}{r}+\frac{2}{r}\ln(r-1)+\ln\left(\frac{r}{r-1}\right)+\frac{3}{r(r-1)}+\frac{1}{r}\sum_{n=2}^{\infty}(-1)^n\frac{(r-1)^{-n}}{n(n-1)}. \tag{8.5}$$

The various diffusion and heat-conduction parameters have been defined in §§4, 6 and 7.

The flame-velocity formula (8.1) is the final result of the present theory. It shows that there is a perturbation of the flame velocity due to acceleration of the flame front and a perturbation due to curvature effects. The particular combination

$$\left(v'+\bar{u}\frac{d\bar{x}_0}{d\eta}\right)_0$$

that occurs in the curvature term permits a direct interpretation: it is the component of the velocity of fluid relative to the flame front, that is tangential to the instantaneous flame position. This interpretation leads to some interesting observations. Consider a flame front of circular shape. If the radius is large compared with the flame thickness, we can apply the present theory locally at every point of the flame. Since the fluid motion is in this case strictly in the radial direction, it is easily found that

$$\left(v'+\bar{u}\frac{dx_0}{d\eta}\right)_0 = 0$$

at every point of the flame, and thus, that up to the first order in flame thickness there is no effect whatever of the curvature of the flame on its propagation velocity.

Now, real flames are almost never two-dimensional. Fortunately, however, an extension of the present theory to three dimensions is almost trivial. As discussed in Eckhaus (1959), it is easily found that the effects of curvature are additive. Thus, if ζ is the third space direction (in a Cartesian co-ordinate system in which the $\zeta-\eta$ plane is tangential to the flame front at the point that we consider) and if w is the fluid velocity in the ζ-direction, the extension of (8.1) to three dimensions is

$$\left(\frac{u'-\nu\bar{x}_0}{\bar{u}}\right)_0 = -c\left\{\frac{\nu}{\bar{u}_0\tau}\left(\frac{u'-\nu\bar{x}_0}{\bar{u}}\right)_0 A_1+\frac{1}{\tau}\left[\frac{\partial}{\partial\eta}\left(\frac{v'}{\bar{u}_0}+\frac{\partial\bar{x}_0}{\partial\eta}\right)_0+\frac{\partial}{\partial\zeta}\left(\frac{w'}{\bar{u}_0}+\frac{\partial\bar{x}_0}{\partial\zeta}\right)_0\right]A_2\right\}+O(\epsilon^2). \tag{8.6}$$

It follows now that for a spherical flame front there will be no effects of curvature on the flame velocity.

Consider finally the flame on a Bunsen-burner, which is nearly conical in shape. If we apply (8.6) at an arbitrary point of the flame, we find that the curvature, caused by the rotational symmetry of the flame, has no effect on the propagation velocity. The only effect that arises is that due to curvature in the meridian plane. Since this curvature is extremely small, except in the tip-region, the total effect of curvature for a Bunsen-burner flame will be extremely small.

The above holds for the spherical flame front quite independently of the magnitude of the constant c. For the Bunsen-burner flame it holds for values of c up to say $c = O(\tau)$. It follows that c can be quite large, and yet its effect will not be felt in most experimental arrangements.

It is somewhat unfortunate for the present theory that the constant c of (8.1) and (8.6) cannot be exactly determined. It is true that in §5 we have found an approximate expression (5.3) for c. However, not much is gained by expressing c in terms of α^*, since this last quantity is a hypothetical constant rate of reaction which cannot easily be correlated with the true physical properties of the mixture. To be realistic we must accept that the present model of combustion does not determine the constant c. From §5 we can conclude only that c is positive, and that it can be large if the ratio of thickness of the burning zone to that of the pre-heat zone is sufficiently small (which is the basic postulate of our model).

9. Application to the problem of stability

We shall now consider the stability of a plane flame front. Let the initial disturbance be sinusoidal in the η-direction: thus

$$\bar{x}_0 = \xi_0 \cos\left(\frac{2\pi}{l}\eta\right). \tag{9.1}$$

We recall that, according to §3 we have

$$x_0(y,t) = \bar{x}_0(\eta)\, e^{\nu t}, \tag{9.2}$$

where x_0 is the instantaneous distance from the flame front to its undisturbed position.

For the stability analysis, solutions of the fluid perturbation velocities in the two regions ahead and behind the flame front are needed, together with boundary conditions at the flame front. These last ones are supplied by the requirement that mass and momentum of the fluid are preserved when passing through the flame front. Details of the derivation can be found elsewhere (Eckhaus 1959;

Emmons 1958). The results for the fluid velocities at the upstream flame boundary are

$$\left(\frac{u'-\nu\bar{x}_0}{\bar{u}}\right)_0 = -\frac{\xi_0}{2(r+\omega)}\left\{\omega^2\frac{r+1}{r}+2\omega-(r-1)\right\}\frac{2\pi}{l}\cos\left(\frac{2\pi}{l}\eta\right), \tag{9.3}$$

$$\frac{\partial}{\partial\eta}\left[\frac{v'}{\bar{u}}+\frac{d\bar{x}_0}{d\eta}\right] = \frac{\xi_0}{2(r+\omega)}\left\{\omega^2\frac{r-1}{r}+2r\omega+(3r-1)\right\}\left(\frac{2\pi}{l}\right)^2\cos\left(\frac{2\pi}{l}\eta\right), \tag{9.4}$$

where

$$\omega = \frac{\nu l}{2\pi\bar{u}_0}.$$

When Landau's hypothesis—that the flame velocity is not affected by the perturbations—is applied, we must put (9.3) equal to zero.

In the present case we introduce (8.1) for the flame velocity and after elimination obtain

$$\omega^3\frac{r+1}{r}\frac{c}{2\pi l\tau}A_1+\omega^2\left[\frac{r+1}{r}+\frac{c}{2\pi l\tau}\left(2A_1+\frac{r-1}{r}A_2\right)\right]$$
$$+\omega\left[2+\frac{c}{2\pi l\tau}\left(-\frac{r-1}{r}A_1+2A_2\right)\right]-(r-1)\left[1-\frac{3r-1}{r-1}\frac{c}{2\pi l\tau}A_2\right]=0. \tag{9.5}$$

We shall now investigate conditions for stability. If the flame is to be stable, the real parts of all roots of (9.5) must be negative. This puts requirements on the coefficients of (9.5). For stability we must have

$$\left[\frac{3r-1}{r-1}\frac{A_2}{A_1}-\frac{2\pi l\tau}{cA_1}\right]>0, \tag{9.6}$$

$$\left[r\left(2\frac{A_2}{A_1}-\frac{r-1}{r}\right)+2\frac{2\pi l\tau}{cA_1}\right]>0, \tag{9.7}$$

$$\left[\frac{r-1}{r}\frac{A_2}{A_1}+2+\frac{r+1}{r}\frac{2\pi l\tau}{cA_1}\right]>0. \tag{9.8}$$

It can easily be shown that the inequalities can be satisfied if and only if A_1 and A_2 are of the same sign. Using definitions (3.2) and remarking that K_3 is a positive number, we arrive at a necessary (though not sufficient) condition for existence of stable solutions

$$\frac{r}{r-1}K_3 \geqslant (K_1+K_2) \geqslant 0. \tag{9.9}$$

Condition (9.9) does not depend on the parameter $l\tau$, which is the ratio of wavelength of disturbance to thickness of the flame front, but only on the properties of the mixture that is being burned. If the mixture is such that condition (9.9) is not satisfied, the flame will be unstable to all disturbances. If, however, the mixture satisfies condition (9.9) there may be stable and unstable regions, depending on the wavelength of the disturbances. For the flame to be stable we find then that the wavelength must satisfy the following conditions

$$\frac{c}{2\pi l\tau}(K_1+K_2) > \frac{r}{3r-1}, \tag{9.10}$$

$$\frac{c}{2\pi l\tau}(r-1)\left[K_3-\frac{3r-1}{r}(K_1+K_2)\right] < 2. \tag{9.11}$$

If the mixture is such that $K_3 < \{(3r-1)/r\}(K_1+K_2)$, then condition (9.11) is trivial and all wavelengths smaller than the value given by (9.10) are stable. If, however, $K_3 > \{(3r-1)/r\}(K_1+K_2)$ a second region of instability will occur at small wavelengths. For sufficiently small (K_1+K_2) the two regions may overlap, so that, after all, the flame will be unstable for all wavelengths.

We shall now illustrate the above discussion with some numerical examples. In order to evaluate K_1, K_2, and K_3, various parameters describing the physical properties of the mixture must be known. In the following we shall use a range of values of these parameters, which on the basis of data of Hirschfelder *et al.* (1954) are estimated to be reasonably likely to occur. It should be stressed that the examples are intended as an illustration of the various situations which can occur, rather than study of some particular fuel-oxidizer mixture.

We take in the present examples $\tau = 100\,\text{cm}^{-1}$ and $r = 5$. For a range of values of μ_1, we then find the following:

μ_1/τ	1·5	2·0	3·0	4·0
K_1	0·88	1·25	1·56	1·70

Consider now K_2. Its value depends on τ as well as on various coefficients of diffusion. If we write it in the form

$$K_2 = -\overline{\left(\frac{n_3}{n}\right)_f}\left(1-\frac{D_{12}}{D_{13}}\right)\bar{K}_2, \tag{9.12}$$

then $\bar{K}_2$ is a function of μ_1/τ and μ_3/μ_1. For this last parameter we write, according to definitions (6.1),

$$\frac{\mu_3}{\mu_1} = \frac{D_{12}}{D_3},$$

where D_{12} is the binary diffusion coefficient for the fuel–oxygen diffusion and D_3 is the effective diffusion coefficient for the diffusion of products of combustion into the unburned mixture. In general, the mean molecular weight of the products of combustion will be about the same as that of oxygen, while the mean molecular weight of the combustible mixture will be higher than that of oxygen. From the behaviour of the coefficients of diffusion with respect to molecular weight we then conclude that

$$\frac{D_{12}}{D_3} < 1.$$

For some representative values we find the following:

	$D_{12}/D_3 = \frac{2}{3}$			$D_{12}/D_3 = \frac{1}{2}$	
μ_1/τ	1·5	2·0	3·0	2·0	3·0
$\bar{K}_2$	2·62	1·80	0·94	2·50	1·40

Finally, the coefficient K_3 is a function only of r, and for $r = 5$ its value is

$$K_3 = 1\cdot 54.$$

We consider now a lean mixture. According to §3, the meanings of the diffusion coefficients are in this case

D_{12} = diffusion of fuel in oxygen,

D_{13} = diffusion of fuel in the products of combustion.

But since the molecular weights of oxygen and the products of combustion are not too different, we conclude that

$$\frac{D_{12}}{D_{13}} \doteqdot 1,$$

which means that K_2 is very small. Applying now the stability conditions we find that the flame is stable if the wavelength of disturbance is smaller than the value given in the following table:

μ_1/τ	1·5	2·0	3·0	4·0
l in cm	$0{\cdot}15c$	$0{\cdot}22c$	$0{\cdot}28c$	$0{\cdot}30c$

We turn now to the rich mixture. Here the meaning of the symbols is

D_{12} = diffusion of oxygen in fuel,

D_{13} = diffusion of oxygen in the products of combustion,

from which follows that

$$\frac{D_{12}}{D_{13}} < 1 \quad \text{and thus} \quad K_2 < 0.$$

Choosing $D_{12}/D_{13} = 0{\cdot}55$, and also taking $\overline{(n_3/n)_f} = 0{\cdot}9$, we now find the following behaviour:

μ_1/τ	Behaviour for $D_{12}/D_3 = \frac{2}{3}$	Behaviour for $D_{12}/D_3 = \frac{1}{2}$
1·5	Unstable for all wavelengths	—
2·0	Stable for $0{\cdot}01c \leqslant l \leqslant 0{\cdot}09c$	Unstable for all wavelengths
3·0	Stable for $l \leqslant 0{\cdot}20c$	Stable for $l \leqslant 0{\cdot}17c$

If we now consider some specific fuel and oxidizer and wish to compare the behaviour of the flame, for lean and rich mixtures, then μ_1/τ is approximately the same for both cases, so that we must compare the above results for a fixed value of this parameter. It is clear, then, that a situation can easily arise in which the lean and the rich flames will behave very differently. For instance, for μ_1/τ up to 2·0, the lean flames will have a range of stable wavelengths, while the rich flames will be unstable to practically all wavelengths.

The stability boundary for lean flames (and for rich flames, if there is one) still depends on the proportionality constant c. According to our combustion model there is some reason to believe that c will be large. If, for instance, $c = 10$, we find the stability boundary for lean flames at wavelengths of 2 to 3 cm. However, no matter how large c is, according to the present results, instability will always occur at sufficiently large wavelengths. This seems to be in disagreement with Markstein's observations; but at this point we must realize that the theory assumes the flame to be of infinite extent, or, what is more realistic, that the wavelength of disturbance is assumed to be small compared with the dimensions of the flame container. In Markstein's experiments the flame was contained in a circular tube of 10 cm diameter. Most probably, for wavelengths of 2 to 3 cm, wall effects can no longer be neglected. Markstein observed that the overall appearance of the cellular flames was different at different wall temperatures, which suggests that the wall effects can be very important.

It should finally be remarked that for large wavelengths gravitational effects must be taken into account.

10. Conclusions

We have seen in the last section that application of the present theory to the problem of stability leads to results which can be interpreted to be in qualitative agreement with the experimentally observed behaviour. Therefore it appears that the essential features of the phenomena have been retained to some degree, in spite of the over-simplification of the theory.

An improvement of the theory can presumably be achieved by introducing a more correct model of the flame structure. The present method of integrating the equations will then probably still be useful. However, if the model is more complex, the increase of labour can be very large. Moreover, progress in this direction is made difficult because of the absence, at present, of a definite theory for one-dimensional, stationary flame propagation. This makes the choice of the model for the perturbation theory almost a matter of taste. As we have already remarked, the present theory can be considered to be a counterpart of the simplest available stationary one-dimensional theory. It cannot be expected to provide all the answers; it may even be incorrect at some points. It will, however, serve its purpose if it does provide the first step in a correct direction.

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